Periodic orbit spectrum in terms of Ruelle–Pollicott resonances

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Abstract

Fully chaotic Hamiltonian systems possess an infinite number of classical solutions which are periodic, e.g. a trajectory "p" returns to its initial conditions after some fixed time τ_p. Our aim is to investigate the spectrum $\{\tau_1, \tau_2, \ldots\}$ of periods of the periodic orbits. An explicit formula for the density $\rho(\tau) = \sum_p \delta(\tau - \tau_p)$ is derived in terms of the eigenvalues of the classical evolution operator. The density is naturally decomposed into a smooth part plus an interferent sum over oscillatory terms. The frequencies of the oscillatory terms are given by the imaginary part of the complex eigenvalues (Ruelle–Pollicott resonances). For large periods, corrections to the well–known exponential growth of the smooth part of the density are obtained. An alternative formula for $\rho(\tau)$ in terms of the zeros and poles of the Ruelle zeta function is also discussed. The results are illustrated with the geodesic motion in billiards of constant negative curvature. Connections with the statistical properties of the corresponding quantum eigenvalues, random matrix theory and discrete maps are also considered. In particular, a random matrix conjecture is proposed for the eigenvalues of the classical evolution operator of chaotic billiards.

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1 Introduction

In chaotic Hamiltonian systems, the unstable classical periodic orbits form a set of measure zero among all the possible trajectories. However, as has been emphasized many times, the periodic orbits are of great interest. In particular, they are very important in the study of the structure of the phase space dynamics and the transport properties of the motion. Another relevant aspect of these orbits is their temporal behavior. At a given energy, the periods τ_p of all the periodic orbits p form a discrete set, $\{\tau_p\} = \{\tau_1, \tau_2, \ldots, \tau_i, \ldots\}$. Many properties of this sequence are of interest in, e.g., understanding the quantum mechanical behavior of chaotic systems. For example, semiclassical theories establish a link between the statistical properties of the quantum eigenvalues and those of the classical periods $\{\tau_p\}$. One is led to answer questions like how grows the number of periodic orbits with increasing period, or what are the correlations (if any) between the periods of different orbits. In this respect, as regards the first question, one of the main results in the field is the exponential growth of their number with increasing period [1]. Concerning the second, there is no definite answer, but semiclassical arguments based on random matrix theory (RMT) suggest that there exists correlations between the orbits (orbits with similar period repel each other) [2].

Our purpose is to further explore the properties of the spectrum of the periods $\{\tau_p\}$ of the periodic orbits of fully chaotic systems. We restrict the index "p" to the primitive orbits only, i.e. repetitions of a given orbit are excluded. Our main result is an explicit formula that relates the density of periods of primitive orbits,

$$\rho(\tau) = \sum_p \delta(\tau - \tau_p) \ , \tag{1}$$

to the eigenvalues of the classical evolution operator, the so–called Ruelle–Pollicott resonances. These resonances, which generically are defined by a set of complex numbers (denoted $\{\gamma\}$), characterize the decay of correlations in the time evolution of phase space densities [3], and provide important information about the transport properties of the system. For an introduction see [4, 5].

The set of periods $\{\tau_p\}$ can thus be explicitly related to another, more fundamental, set, the Ruelle–Pollicott resonances $\{\gamma\}$. In this way, the properties of the τ_p's can directly be related to the properties of the γ's. The results allow, in particular, to make a systematic analysis of the structure of the spectrum of periods, focusing from the larger towards the smaller scales.

Semiclassical theories à la Gutzwiller or Balian–Bloch [6, 7] describe the density of quantum states in terms of the periodic orbits of the corresponding classical system. Here, in turn, we establish a connection between the density of periods of the periodic orbits and the Ruelle–Pollicott resonances, i.e. the eigenvalues of the classical evolution operator. In this way, our results allow to establish a link between

the eigenvalues of the quantum and classical evolution operators that, hopefully, will be useful to more clearly elucidate their properties and correspondences.

Not much is known at present about the distribution in the complex plane of the eigenvalues of the classical evolution operator of chaotic Hamiltonian systems (see, e.g., [4]). Paraphrasing Alfredo Ozorio de Almeida [8], it is fair to say that though of great theoretical interest, the formula to be derived below merely relates our ignorance of the periodic orbit spectrum to the even more mysterious maze of the eigenvalues of the classical evolution operator. However, there is an increasing effort to understand the properties and physical interpretation of the latter, and their study is a central theme in several of the most interesting recent developments in classical chaotic systems, and of their quantum counterparts. For instance, in quantum systems the Ruelle–Pollicott resonances lurk behind many interesting effects. They provide the most simple explanation of the long–range non universal correlations observed in the quantum spectra of bounded Hamiltonian systems [9, 10], and show up in experiments that measure the transmission through open microwave chaotic cavities [11]. The spectrum $\{\gamma\}$ is also a central issue in recent field theoretic approaches whose aim is to demonstrate the validity of RMT in chaotic systems [12], and appears in some mathematical models of quantum chaos, the Riemann zeros [13]. More recently, several studies have clarified the correspondence between the quantum and classical propagators for discrete maps in the presence of noise [14].

The starting point of our study is an expression of the trace of the evolution operator as a sum over the periodic orbits (§3). An inversion of that formula, based on the Möbius inversion technique, is implemented. Assuming that the spectrum of the evolution operator consists of isolated resonances, the inversion leads to a general formula for the density $\rho(\tau)$ in terms of the eigenvalues $\{\gamma\}$ (§4). It is also shown, in the same section, that a natural decomposition of the density emerges, where resonances located on the real axis determine the average or smooth behavior of the density, while oscillatory interferent terms arise from the complex resonances. An alternative and mathematically more accurate formula for $\rho(\tau)$, based on the Ruelle zeta function (instead of the determinant of the evolution operator), is first presented in §2. It serves as a reference for the calculations of §4, and complements the results obtained. Both approaches are compared in §4. Two illustrative examples are worked out in §5. The first one is the geodesic motion in a billiard of constant negative curvature. The spectrum of the evolution operator can be explicitly computed in this case. This allows to write down a formula for the density of periods of the periodic orbits, thus illustrating the general approach of §4. The results reproduce, in our formalism, those of Ref. [15]. The second example is based on the Riemann zeta function. Though the dynamical basis for this model is hypothetical, it is included here mainly to stress the existing analogies and similarities with known results in analytic number theory. Finally, §6 contains some general remarks and conjectures concerning the spectrum of the evolution operator, inspired by the

results of §5 and quantum chaos theory. Special emphasis is put in the connection with the statistical fluctuations of quantum eigenvalues and random matrix theory. We also show that important qualitative differences exist between the spectrum of the classical evolution operator of smooth flows and that of discrete maps.

2 The density of periods of periodic orbits: Ruelle zeta

In this section the aim is to derive an explicit formula for the density of periods of the primitive periodic orbits, Eq.(1). The density will be expressed in terms of the zeros and poles, located in the complex plane, of a particular function, the Ruelle zeta function. To simplify, we will from now on make reference to the zeros and poles of the Ruelle zeta as its "singularities". The calculations presented in this section serve as a basis for those of §4, where the density $\rho(\tau)$ will be expressed in terms of the eigenvalues of the evolution operator.

The starting point is the function

$$P(\tau) = \sum_p \sum_{r=1}^{\infty} \tau_p \ \delta(\tau - r\tau_p) \ . \tag{2}$$

The index r accounts for the repetitions (or multiple traversals) of a given primitive periodic orbit "p". The function $P(\tau)$ is naturally associated with a zeta function. To see this, we reproduce a well known derivation [4], which consists in including inside the double sum (2) a factor $\exp[b(\tau - r\tau_p)]$, whose value is one because of the delta function (b is a real positive constant). Then, expressing the delta function as $\delta(\tau - r\tau_p) = (2\pi)^{-1} \int_{-\infty}^{\infty} \exp[i\xi(\tau - r\tau_p)] d\xi$, and making the change of variables $\xi = -i\omega$, $P(\tau)$ takes the form

$$P(\tau) = \frac{\mathrm{e}^{b\tau}}{2\pi i} \sum_p \sum_{r=1}^{\infty} \tau_p \ \mathrm{e}^{-br\tau_p} \int_{-i\infty}^{i\infty} \mathrm{e}^{\omega(\tau - r\tau_p)} d\omega \ .$$

For b positive and sufficiently large, the sum is convergent, and can therefore be interchanged with the integral. Doing that, and making the additional change of variables $s = \omega + b$, we now get,

$$P(\tau) = -\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} ds \ \mathrm{e}^{s\tau} \frac{\partial}{\partial s} \sum_p \sum_{r=1}^{\infty} \frac{\mathrm{e}^{-sr\tau_p}}{r} \ .$$

Finally, using the expansion $\log(1-x) = -\sum_{n=1}^{\infty} x^n/n$, we obtain the relation,

$$P(\tau) = -\frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} ds \ \mathrm{e}^{s\tau} \frac{\partial}{\partial s} \log Z_R(s) \ , \tag{3}$$

where $Z_R(s)$ is the topological or Ruelle zeta function [16, 17],

$$Z_R(s) = \prod_p \left(1 - \mathrm{e}^{-s\tau_p}\right)^{-1} \ , \tag{4}$$

for $\mathrm{Re}(s)$ large.

Equation (3) allows to make a connection between the sum (2) and the analytic properties of $Z_R(s)$. In order to proceed we thus need some information about the latter. In fact, for certain classes of hyperbolic systems, it can be shown that $Z_R(s)$ is analytic for $\mathrm{Re}(s) > h_{\mathrm{top}}$, where h_{top} is the topological entropy, and has a meromorphic extension to the whole complex plane [17]. Restricting our analysis to this case, from (3) an explicit and simple formula follows (with $b > h_{\mathrm{top}}$, and closing the path of integration towards the left part of the complex plane). It expresses $P(\tau)$ in terms of the location in the complex plane, denoted η, of the singularities of $Z_R(s)$,

$$P(\tau) = \sum_\eta g_\eta \ \mathrm{e}^{\eta\tau} \ . \tag{5}$$

In this equation the integer g_η is the multiplicity, and is positive for poles and negative for zeros. The index η has a double significance: it serves as an index to enumerate the singularities, and also denotes their location in the complex plane, at $s = \eta$.

Two alternative and distinct formulas for $P(\tau)$ are thus available, one as a sum over the periodic orbits, Eq.(2), the other as a sum over the singularities of $Z_R(s)$, Eq.(5). It is convenient to rewrite Eq.(2) in terms of the density, using the definition (1),

$$P(\tau) = \tau \sum_{r=1}^{\infty} \frac{1}{r^2} \ \rho(\tau/r) \ . \tag{6}$$

From now on, the idea of the computation is the following. If we manage to invert equation (6), and express the density $\rho(\tau)$ in terms of $P(\tau)$, we are done, because Eq.(5) can then be used for $P(\tau)$, and the final output would be an expression of $\rho(\tau)$ in terms of the singularities η of the function $Z_R(s)$.

Inversion problems have many important physical applications, but are in general difficult to solve. In our case, the inversion of Eq.(6) will be based on the Möbius inversion formula. This is a technique developed in the nineteenth century, that has been extensively exploited in number theory [18]. More recently, it has found concrete applications in physics, like for example for computing the phonon density of states from experimental measurements of the specific heat of solids [19].

The inversion proceeds as follows. Consider the sum

$$S_1 = \sum_{m=1}^{\infty} \frac{\mu(m)}{m} P(\tau/m) \ , \tag{7}$$

where $\mu(m)$ is the Möbius function [18]. This is a number–theoretic function, whose properties are based on the prime decomposition of the integer m. It is defined as,

$$\mu(m) = \begin{cases} 1 & \text{if } m = 1 \\ (-1)^k & \text{if } m \text{ is a product of } k \text{ distinct primes} \\ 0 & \text{if } m \text{ has one or more repeated prime factors ,} \end{cases}$$

(and thus $\mu(m) = 1, -1, -1, 0, -1, +1, \ldots$ for $m = 1, 2, 3, 4, 5, 6, \ldots$, a quite erratic function). If, in Eq.(7), we use for P the series defined by the r.h.s. of Eq.(6), we obtain

$$S_1 = \tau \sum_{m,r=1}^{\infty} \frac{\mu(m)\ \rho(\tau/rm)}{(rm)^2} = \tau \sum_{n=1}^{\infty} \sum_{m/n} \frac{\mu(m)\ \rho(\tau/n)}{n^2} = \tau \rho(\tau)\ ,$$

where the sum $\sum_{m/n}$ runs over the divisors m of n. The last equality is the key point of the inversion technique. It follows from a remarkable property of the Möbius function, namely $\sum_{m/n} \mu(m) = \delta_{n,1}$. Finally, combining the last equation with Eq.(7), we obtain an equation for the density,

$$\rho(\tau) = \frac{1}{\tau} \sum_{m=1}^{m_c} \frac{\mu(m)}{m} P(\tau/m) = \frac{1}{\tau} \sum_{m=1}^{m_c} \frac{\mu(m)}{m} \sum_{\eta} g_\eta\ \mathrm{e}^{\eta\tau/m}\ . \tag{8}$$

The sum over m has been truncated at a value equal to the integer part of τ/τ_{min}, where τ_{min} is the period of the shortest periodic orbit of the system,

$$m_c = [\tau/\tau_{min}]\ .$$

This truncation is a consequence of the fact that $P(\tau) = 0$ for $\tau < \tau_{min}$ (cf Eq.(2)), and therefore $P(\tau/m) = 0$ for $m > m_c$ in Eq.(8).

Equation (8) is our first result. It expresses the density of periods of the primitive periodic orbits in terms of the singularities of the Ruelle zeta function.

The right hand side of Eq.(8), should, in principle, reproduce a series of delta peaks located, according to the definition of $\rho(\tau)$, at the periods of the primitive periodic orbits. In order to better display the structure of Eq.(8), and see the correspondence with the series of delta peaks, it is useful to consider a decomposition of the density in two parts,

$$\rho(\tau) = \overline{\rho}(\tau) + \widetilde{\rho}(\tau)\ . \tag{9}$$

This decomposition is associated to a classification of the resonances into two distinct sets, each contributing to one of the two terms in the r.h.s. of Eq.(9). Because $P(\tau)$ is a real function, the Ruelle zeta satisfies a simple functional equation (which follows from Eq.(3)),

$$\left(Z_R(s)\right)^* = Z_R\left(s^*\right)\ ,$$

(the star denotes complex conjugate). A singularity η of $Z_R(s)$ is therefore either simple and real, or complex and comes in conjugate pairs symmetric with respect to the real axis. This property is at the origin of the decomposition (9), with the real singularities contributing to $\overline{\rho}(\tau)$, and the complex ones to $\widetilde{\rho}(\tau)$. If the location of the resonances in the complex plane is written as

$$\eta = q_\eta \pm i t_\eta \ ,$$

with q_η real and t_η real positive, then the two contributions to the density can be expressed as,

$$\overline{\rho}(\tau) = \frac{1}{\tau} \sum_{m=1}^{m_c} \frac{\mu(m)}{m} \sum_{\eta \in \Re} g_\eta \ \mathrm{e}^{q_\eta \tau / m} \ , \tag{10}$$

and

$$\widetilde{\rho}(\tau) = \frac{2}{\tau} \sum_{m=1}^{m_c} \frac{\mu(m)}{m} \sum_{\eta, \ t_\eta > 0} g_\eta \ \mathrm{e}^{q_\eta \tau / m} \cos(t_\eta \tau / m) \ , \tag{11}$$

where in Eq.(10) the sum is made over the real singularities of $Z_R(s)$, whereas in Eq.(11) it is made over the complex ones located in the upper half part of the complex plane.

The first contribution, $\overline{\rho}(\tau)$, is given by a sum of real exponential terms. It has therefore a smooth dependence on τ, and describes the average properties of the density. It reproduces the behavior of the singular sum (1) when $\rho(\tau)$ is smoothed on a scale which is large compared to the average spacing between delta peaks. To reproduce the average part of the density of periods we thus need to know the location of the real singularities. It is known, in particular, that in hyperbolic systems its rightmost real singularity, which controls the asymptotic growth when $\tau \to \infty$, is a simple pole at $s = h_{\rm top}$ [1],

$$\eta_0 = q_0 = h_{\rm top} \ ; \qquad g_{\eta_0} = 1 \ . \tag{12}$$

The presence of this pole implies, keeping the term $m = 1$ in Eq.(10),

$$\overline{\rho}_0(\tau) \overset{\tau \to \infty}{=} \frac{\mathrm{e}^{h_{\rm top} \tau}}{\tau} \ . \tag{13}$$

We thus recover, to leading order, the well known exponential growth of periodic orbits in chaotic systems, with the typical rate given by the inverse of the topological entropy [1]. However, the inversion procedure employed here allows to go beyond that result, and obtain subdominant corrections to the average growth of the density. It is remarkable, indeed, that the contribution of the pole at $s = h_{\rm top}$ gives also, from Eq.(10), a series of exponentially large corrections to the asymptotic behavior, whit signs depending on the Möbius function,

$$\overline{\rho}_0(\tau) = \frac{1}{\tau} \sum_{m=1}^{m_c} \frac{\mu(m)}{m} \ \mathrm{e}^{h_{\rm top} \tau / m} = \frac{\mathrm{e}^{h_{\rm top} \tau}}{\tau} - \frac{\mathrm{e}^{h_{\rm top} \tau / 2}}{2\tau} - \frac{\mathrm{e}^{h_{\rm top} \tau / 3}}{3\tau} - \ldots \ . \tag{14}$$

The first corrections lower the density, while the first positive one occurs for $m = 6$.

Besides the pole at $h_{\rm top}$, other real singularities of $Z_R(s)$, with real part $q_\eta < h_{\rm top}$, add further sub–leading corrections to the density of periods. Those located in the negative part of the real axis contribute with exponentially small corrections. We are not aware of any generic result about the location in the complex plane of the additional real singularities, and how their contributions compare to the corrections that come from η_0.

The remaining contribution to the density, $\widetilde{\rho}(\tau)$, behaves quite differently. This term is responsible for the discrete nature of the spectrum of periods of the periodic orbits. On top of the average behavior $\overline{\rho}$, each complex singularity of $Z_R(s)$ adds an oscillatory term to the density of periods. It is the interference of the oscillatory contributions of all the complex singularities that, formally, reproduces the delta–peak structure of Eq.(1).

The amplitude of each oscillatory term is given by the exponential of the real part q_η of the singularity. In contrast, the inverse of the imaginary part, $2\pi m t_\eta^{-1}$, is the period of the oscillation. Singularities with the smaller imaginary part describe long range fluctuations with respect to the smooth behavior of the density, on scales which may be much larger than the typical time that separates two neighboring orbits (terms with $m > 1$ are of longer range, but their weight is of lower order in the limit $\tau \to \infty$). As singularities with increasing imaginary part are included, details of $\rho(\tau)$ on smaller scales are resolved. For instance, to distinguish individual peaks of $\rho(\tau)$ located around τ requires complex singularities with imaginary part of the order of the average density of periods, $t_\eta \sim {\rm e}^{h_{\rm top}\tau}/\tau$.

Eq.(8) provides therefore a harmonic decomposition of the density of periods, were the frequency of each sinusoidal wave is given by a fraction of the imaginary part of a complex singularity. Since arbitrary high frequencies are needed to reproduce a delta peak, Eq.(8) also implies that, generically, singularities with arbitrarily large imaginary part should exist. But we have no additional information about their distribution in the complex plane.

Eq.(8) can be integrated with respect to τ to obtain a harmonic formula for the cumulative distribution, or counting function of the periods. This function is defined as the number of primitive periodic orbits whose period is smaller than τ,

$$N(\tau) = \int_0^\tau \rho(x)\ dx = \sum_p \Theta(\tau - \tau_p)\ , \tag{15}$$

where Θ is Heaviside's step function. The result of the integration is,

$$N(\tau) = \sum_{m=1}^{m_c} \frac{\mu(m)}{m}\ \sum_\eta g_\eta\ {\rm Ei}\left(\frac{\eta\tau}{m}\right)\ , \tag{16}$$

where Ei is the exponential integral function. Similarly to Eq.(8), this expression

is, formally, exact. The same decomposition and general remarks as for the density apply to this function.

3 The trace of the evolution operator

In the previous section, we described the periodic orbit density and the counting function in terms of the singularities of the Ruelle zeta function. In this and the next section we will introduce an alternative description of the density, based on the eigenvalues of the classical evolution operator. Our motivations for doing this are the following. On the one hand, an explicit relation between the periodic orbits and the eigenvalues of the evolution operator is of clear theoretical interest. It helps in understanding the properties of both sets, as well as their correspondences. On the other hand, the connection is a central issue in the study of the quantum and classical behavior of chaotic systems and allows, via semiclassical techniques, to relate the quantum and classical eigenvalues.

Before presenting in §4 the derivation of the alternative description of the density of periods, we need to previously introduce some definitions and basic equations related to the classical evolution operator.

Consider a classical system whose dynamical state is defined by the phase–space coordinate x. After a time τ, the point x evolves into a new state $y = f_\tau(x)$. The kernel of the evolution operator of the deterministic motion is defined by a Dirac distribution,

$$\mathcal{L}_\tau(y, x) = \delta(y - f_\tau(x)) \ . \tag{17}$$

We shall assume throughout the paper that $f_\tau(x)$ describes the conservative smooth flow of a closed Hamiltonian system. Open systems are not treated here, but may be considered as well within the same formalism [4].

The trace of the evolution operator is defined as

$$R(\tau) = \int dx \ \mathcal{L}_\tau(x, x) \ , \tag{18}$$

where x is integrated over the phase space. The function $\mathcal{L}_\tau(x, x)$ is the conditional probability density for the system to be at the point x by the time τ if the initial state was at the same point. $R(\tau)$ is therefore proportional to the classical return probability at time τ. The trajectory starting at x and coming back to the same point after a time τ defines a closed loop in phase space. Any phase–space closed loop defines a cyclic motion, because the system returns to its initial conditions. For a fully chaotic dynamics, an explicit expression of $R(\tau)$ in terms of the periodic orbits of the system was obtained in Ref. [20],

$$R(\tau) = \sum_p \sum_{r=1}^{\infty} \frac{\tau_p}{|\det(M_p^r - 1)|} \ \delta(\tau - r\tau_p) \ . \tag{19}$$

The sum is made over the primitive periodic orbits p and over their repetitions r. Each orbit has a period τ_p and a monodromy matrix M_p. The latter describes the stability of the orbit. The factor $|\det(M_p^r - 1)|^{-1}$ is related to the overlap between an initial cloud, centered initially around the orbit p, and its iterate, after a time $\tau = r\tau_p$, in a linear approximation. The function $R(\tau)$ has therefore a peak at the period of each primitive periodic orbit, or at one of its repetitions, with a weight inversely proportional to its stability.

Notice that Eq.(19) is similar to Eq.(2), with the important difference of the stability factor in the denominator in the former. In spite of this difference, similar steps as those who led in the previous section from Eq.(2) to Eq.(3) can be followed for $R(\tau)$. They lead to a connection between the trace of the evolution operator and a zeta function. We will not repeat them here, but only mention that, for reasons to be understood below, the sign in the change of variables between ξ and ω is the opposite here. The result, analogous to Eq.(3), is,

$$R(\tau) = -\frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} ds \ \mathrm{e}^{-s\tau} \frac{\partial}{\partial s} \log Z(s) \ , \tag{20}$$

where a is a real constant < 0. The function $Z(s)$ is called the spectral determinant, or Smale zeta function, and is defined as [4, 16],

$$Z(s) = \exp\left[- \sum_{p,r} \frac{\mathrm{e}^{sr\tau_p}}{r|\det(M_p^r - 1)|} \right] \ . \tag{21}$$

The complex variable s has units of inverse of time. Eq.(20) relates the trace of the evolution operator to the analytic structure of the function $Z(s)$. Under some general conditions valid for a certain class of hyperbolic systems (and analogous to those assumed in the previous section for $Z_R(s)$), the Smale zeta function is generically an entire function [21, 22] (i.e., analytic at all finite points in the complex plane). More complicated analytic structures of $Z(s)$ may arise, like for example branch cuts that lead to power law decays in intermittent systems [9]. We will ignore here these other possibilities. Assuming, therefore, that $Z(s)$ is entire, from Eq.(20) the trace of the evolution operator can be expressed as [20],

$$R(\tau) = \sum_{\gamma} g_\gamma \ \mathrm{e}^{-\gamma\tau} \ , \tag{22}$$

where the γ's are the zeros of $Z(s)$. The complex set of points $\{\gamma\}$ define the (complex and discrete) spectrum of the evolution operator. In this equation and in the rest of the paper, the index γ has a double significance: it serves as an index to enumerate the zeros, and also denotes their location in the complex plane, at $s = \gamma$.

The spectrum $\{\gamma\}$, given by the zeros of the entire function $Z(s)$, characterizes the relaxation towards equilibrium of classical statistical ensembles [3]. The complex

zeros are usually called Perron–Frobenius or Ruelle–Pollicott resonances. We shall, however, often refer to the *whole* set of zeros (real and complex) as *resonances*.

The positive index g_γ in (22) is the multiplicity of the resonance. The physically relevant modes being decaying ones, the corresponding resonances lie in the positive half plane $\mathrm{Re}(\gamma) \geq 0$ (this justifies the choice of sign in the derivation of Eq.(20)). It is in general a difficult problem to determine analytically the spectrum γ for a particular system. However, in Hamiltonian systems, where the energy is conserved, there exists a well defined long–time equilibrium state, given by the invariant measure on the energy shell with the microcanonical weight. The existence of this equilibrium state is manifested in the analytic properties of $Z(s)$ by the presence of a simple "ergodic" zero [5], located at

$$\gamma_0 = 0 \ , \qquad\qquad g_{\gamma_0} = 1 \ . \tag{23}$$

The ergodic zero, located at the origin, corresponds to the unique invariant measure (in the sense of statistical ensembles) in fully chaotic systems. From Eq.(22) it follows that this very general property implies that $\lim_{\tau\to\infty} R(\tau) \to 1$, which expresses the equiprobability to find the system at any phase space point. This fixes the normalization of the trace of the evolution operator, or "return probability" (the volume is set to one). Other resonances, with $\mathrm{Re}(\gamma) > 0$, are associated to decaying modes that describe the process of relaxation of an initial cloud towards equilibrium.

4 The density of periods of periodic orbits: Ruelle–Pollicott resonances

The two expressions of the trace introduced in the previous section, Eqs.(19) and (22), relate a sum over the eigenvalues of the evolution operator to a sum over all the periodic orbits of the system. Similarly to §2, we can exploit this connection to derive an explicit formula for the density of periods of primitive periodic orbits, $\rho(\tau)$, but now expressed in terms of the eigenvalues of the evolution operator (instead of the singularities of the Ruelle zeta function). This formula will take again the form of a harmonic decomposition, i.e., the density will be expressed as an interferent sum over oscillatory terms. The frequency of oscillation of these terms will be given by the imaginary part of the Ruelle–Pollicott resonances.

To compute $\rho(\tau)$, now the starting point is Eq.(19). To express $R(\tau)$ in terms of the density of periods, the main obstacle are the stability factors, $|\det(M_p^r - 1)|$. These are not only functions of the period of the orbits, but depend also on their stability. However, in fully hyperbolic Hamiltonian systems, where periodic orbits are dense in phase space, the stability or Lyapounov exponents of long orbits are the same for almost all orbits [23]. Therefore, for long orbits the stability factor

becomes only a function of the period,

$$| \det(M_p^r - 1)| \overset{\tau \to \infty}{\longrightarrow} | \det(M_{\tau_p}^r - 1)| \ . \tag{24}$$

Within this approximation, and using the definition (1) of the density, the function $R(\tau)$ may be rewritten as,

$$R(\tau) = \tau \sum_{r=1}^{\infty} \frac{1}{r^2 | \det(M_{\tau/r}^r - 1)|} \ \rho(\tau/r) \ .$$

The factor $| \det(M_\tau^r - 1)|$ is in fact a function of the variable $r\tau$ (cf, for instance, Ref. [4]), and therefore $| \det(M_{\tau/r}^r - 1)| = | \det(M_\tau - 1)|$ depends only on the period. The density then takes the form,

$$R(\tau) = \frac{\tau}{| \det(M_\tau - 1)|} \sum_{r=1}^{\infty} \frac{1}{r^2} \ \rho(\tau/r) \ . \tag{25}$$

From now on, the situation is similar to that of §2. We should invert equation (25), and express the density $\rho(\tau)$ in terms of $R(\tau)$. Then, using Eq.(22) for $R(\tau)$, we will get an expression of $\rho(\tau)$ in terms of the eigenvalues γ. It is convenient, to simplify the computation, to define a new function

$$G(\tau) = \frac{| \det(M_\tau - 1)|}{\tau} \ R(\tau) = \sum_{r=1}^{\infty} \frac{1}{r^2} \ \rho(\tau/r) \ . \tag{26}$$

Then, consider the sum,

$$S_2 = \sum_{m=1}^{\infty} \mu(m) G(\tau/m)/m^2 \ ,$$

where we again make use of the Möbius function $\mu(m)$. The sum S_2 is evaluated in two different ways. First, using for G the series expansion in the r.h.s. of Eq.(26). This gives,

$$S_2 = \sum_{m,r=1}^{\infty} \frac{\mu(m) \rho(\tau/rm)}{(rm)^2} = \sum_{n=1}^{\infty} \sum_{m/n} \frac{\mu(m) \rho(\tau/n)}{n^2} = \rho(\tau) \ ,$$

where the sum $\sum_{m/n}$ is made over the divisors m of n. The last equality follows, as in §2, from the property $\sum_{m/n} \mu(m) = \delta_{n,1}$. The second way to evaluate S_2 is using for G the first equality in Eq.(26). Combining both, we obtain,

$$\rho(\tau) = \frac{1}{\tau} \sum_{m=1}^{m_c} \frac{\mu(m)}{m} \ | \det(M_{\tau/m} - 1)| \ R(\tau/m) \ . \tag{27}$$

As in §2, the sum over m has been truncated at a value equal to the integer part of τ/τ_{min}, where τ_{min} is the period of the shortest periodic orbit of the system. This truncation is a consequence of the fact that $R(\tau) = 0$ for $\tau < \tau_{min}$ (cf Eq.(19)), and therefore $R(\tau/m) = 0$ for $m > m_c$ in Eq.(27).

Eq.(27) gives an explicit connection between the density of periods and the trace of the evolution operator. To express the density in terms of the eigenvalues of the evolution operator, we simply use Eq.(22). This gives,

$$\rho(\tau) = \frac{1}{\tau} \sum_{m=1}^{m_c} \frac{\mu(m)}{m} \; |\det(M_{\tau/m} - 1)| \; \sum_{\gamma} g_\gamma \; \mathrm{e}^{-\gamma\tau/m} \; . \tag{28}$$

Equation (28) is our main result. It provides an alternative formula for the density of periods of primitive periodic orbits. The structure of Eq.(28) is very similar to that of Eq.(8), with two important differences: it includes the stability factors, and the sum is made now over the eigenvalues $\{\gamma\}$ of the classical evolution operator (given by the zeros of $Z(s)$), instead of the singularities of $Z_R(s)$.

The right hand side of Eq.(28) should, in principle, also reproduce a series of delta peaks located, according to the definition of $\rho(\tau)$, at the periods of the primitive periodic orbits. Since the Smale zeta function also satisfies the functional equation,

$$(Z(s))^* = Z\left(s^*\right) \; ,$$

the density can be, as in §2, decomposed into two parts, $\rho = \overline{\rho} + \widetilde{\rho}$, where real eigenvalues of the evolution operator contribute to $\overline{\rho}$, and complex symmetric ones to $\widetilde{\rho}$. If the location of the resonances in the complex plane are written as

$$\gamma = q_\gamma \pm i t_\gamma \; ,$$

with q_γ and t_γ real positive (or zero), then the smooth and oscillatory contributions to the density can be express as,

$$\overline{\rho}(\tau) = \frac{1}{\tau} \sum_{m=1}^{m_c} \frac{\mu(m)}{m} \; |\det(M_{\tau/m} - 1)| \; \sum_{\gamma\in\Re^+} g_\gamma \; \mathrm{e}^{-q_\gamma\tau/m} \; , \tag{29}$$

and

$$\widetilde{\rho}(\tau) = \frac{2}{\tau} \sum_{m=1}^{m_c} \frac{\mu(m)}{m} \; |\det(M_{\tau/m} - 1)| \; \sum_{\gamma\in \mathrm{C}^+} g_\gamma \; \mathrm{e}^{-q_\gamma\tau/m} \cos(t_\gamma\tau/m) \; , \tag{30}$$

respectively ($\Re^+$ denotes the positive real axis (including the origin), and C^+ the upper right sector of the complex plane).

How these results compare to Eqs.(10) and (11)? Let's start the comparison with the average part. In the limit $\tau \to \infty$, the leading order contribution to Eq.(29)

is given by the real zero with the smallest real part. That zero is known, it is the ergodic zero denoted γ_0 in the previous section, whose location at the origin is the only generic property known about the spectrum of eigenvalues of the evolution operator in hyperbolic systems. Keeping only the term $q_\gamma = 0$ and $m = 1$ in Eq.(29), we obtain

$$\overline{\rho}(\tau) \overset{\tau\to\infty}{=} \frac{|\det(M_\tau - 1)|}{\tau} \; . \tag{31}$$

Thus, in the limit $\tau \to \infty$, the leading order behavior of the average density of periods is proportional to the stability factor of the orbits. When compared to the leading order obtained in §2 from the leading pole of the Ruelle zeta function, Eq.(13), this result implies,

$$|\det(M_\tau - 1)| \overset{\tau\to\infty}{=} \mathrm{e}^{h_{\rm top}\tau} \; . \tag{32}$$

This correspondence is equivalent to the Hannay–Ozorio de Almeida sum rule, derived in [23,24] from a uniformity principle. It expresses the counterbalance between the exponential proliferation of the periodic orbits and the growth of their (positive) instability. It is also a consequence of Pesin's equality, that relates the topological entropy to the sum of the positive Lyapounov exponents [25]. Here it has an analytic significance, it expresses the correspondence between the pole at $h_{\rm top}$ of the Ruelle zeta function and the ergodic zero at the origin of the Smale zeta function.

The higher order terms $m > 1$ obtain from the ergodic zero in Eq.(29) give for the average density a contribution equivalent to Eq.(14). Besides the ergodic zero, other real resonances $(q_\gamma > 0, t_\gamma = 0)$ of $Z(s)$ add further sub–leading corrections to the density of periods. However, no generic result concerning their location is known.

The remaining contribution to the density, $\widetilde{\rho}(\tau)$, gives the oscillatory terms. In this approach, based on the eigenvalues of the evolution operator, the amplitude of each oscillatory term is determined by a competition between the exponential growth of the stability factor $|\det(M_{\tau/m} - 1)|$ and the exponential decay of $\mathrm{e}^{-q_\gamma \tau/m}$. Using the asymptotic approximation (32) for the determinant, general arguments indicate that the complex resonances with the smaller real part should be located on the strip $0 < q_\gamma < h_{\rm top}$, and therefore that the leading oscillatory contributions have individual amplitudes that grow exponentially in time [26]. The inverse of the complex part, $2\pi m t_\gamma^{-1}$, is the period of the oscillation. Resonances with the smaller imaginary part describe long range fluctuations on top of the smooth behavior of the density, on scales that can be, depending on the value of t_γ, much larger than the typical time that separates two neighboring orbits (terms with $m > 1$ are of longer range, but their weight is of lower order in the limit $\tau \to \infty$). As resonances with increasing imaginary part are included, details of $\rho(\tau)$ on smaller scales are resolved. For instance, to distinguish individual peaks of the density located around a period τ

requires complex resonances with imaginary part of the order of the average density of periods, $t_\gamma \sim e^{h_{\rm top}\tau}/\tau$.

Eq.(28) therefore provides an alternative harmonic decomposition of the density, were the frequency of each sinusoidal wave depends on the imaginary part of the position in the complex plane of the eigenvalues of the evolution operator (Ruelle–Pollicott resonances). Since arbitrary high frequencies are needed to reproduce a delta peak, Eq.(28) implies that, generically, resonances with arbitrarily large imaginary part should exist (similarly to the Ruelle zeta function).

Eq.(28) can be integrated with respect to τ to obtain an alternative formula for the cumulative distribution, or counting function of the periods, Eq.(15). To compute the integral, an explicit form of the determinant is needed, that depends on the dimensionality of the system. However, asymptotically the approximate expression (32) can be used. This yields,

$$N(\tau) \approx \sum_{m=1}^{m_c} \frac{\mu(m)}{m} \; \sum_\gamma g_\gamma \; {\rm Ei}\left[(h_{\rm top} - \gamma) \frac{\tau}{m} \right] \ , \tag{33}$$

which has to be compared to Eq.(16). The same decomposition and general remarks as for the density apply for this function. The main contributions to the smooth part are obtained from the ergodic zero, $\gamma = 0$, and oscillatory terms are given by the complex resonances.

We have therefore obtained two alternative and distinct descriptions of the density of periods of the periodic orbits. From a mathematical point of view, Eq.(8) is preferable to Eq.(28), because in the derivation of the latter we have ignored some possible fluctuations of the stability factors that may occur at short times. However, from a physical point of view the latter description is clearly more interesting, since the Ruelle–Pollicott resonances, or eigenvalues of the classical evolution operator, have a clear and intrinsic physical content, directly related to the classical dynamics. In practice, it is interesting to exploit both approaches. This leads to consider in more detail the relationships between them.

As we have shown, the ergodic zero of $Z(s)$ and the pole of $Z_R(s)$ at $h_{\rm top}$ carry similar (though not exactly equivalent) information. The pole at $h_{\rm top}$ leads to the smooth contribution (14), whereas the ergodic zero generates a series which is identical but with $e^{h_{\rm top}\tau/m}$ replaced by $|\det(M_{\tau/m} - 1)|$. It is only in the asymptotic approximation (32) of the stability factor that their contributions coincide. It is interesting to explore the correspondence between the analytic structure of both functions in the latter approximation. The simplest procedure is to make, in Eq.(21), the replacement $|\det(M_p^r - 1)| \approx e^{h_{\rm top} r \tau_p}$. This approximate Smale zeta function takes the form,

$$Z(s) \approx \exp\left[- \sum_{p,r} \frac{e^{(s-h_{\rm top}) r \tau_p}}{r} \right] = \prod_p \left[1 - e^{(s-h_{\rm top})\tau_p} \right] = Z_R^{-1}(h_{\rm top} - s) \ , \tag{34}$$

where we have used the expansion $\log(1-x) = -\sum_{n=1}^{\infty} x^n/n$ to compute the sum over the repetitions. In this approximation, the poles/zeros η of $Z_R(s)$ are mapped into zeros/poles γ of $Z(s)$, located at $\gamma = h_{\rm top} - \eta$. This correspondence is expected to hold for long times, where the approximation of the stability factor made before is valid. Using the location of the singularities of $Z_R(s)$, the mapping (34) should therefore accurately describe the location in the complex plane of the zeros of $Z(s)$ that are responsible for the asymptotic behavior of the density. Those are the zeros whose real part is small. It maps, in particular, the pole of $Z_R(s)$ at $h_{\rm top}$ into the ergodic zero of $Z(s)$ at the origin. In contrast, resonances with real part far away from the origin need not be in correspondence. Moreover, the approximation (34) produces a function $Z(s)$ which is meromorphic, rather than entire, as assumed before. Globally, the mapping (34) is clearly false.

5 Illustrative examples

The aim now is to find systems were the analytic structure of $Z(s)$ and $Z_R(s)$ can be computed, thus allowing to write–down explicitly the formula for the density of periods of the periodic orbits, and to compare and illustrate both approaches. Another interest of such a study is to gain some insight into the distribution of the resonances and singularities in the complex plane in concrete examples. The explicit computation of the analytic structure of the zeta functions is a quite difficult task in general. However, it is doable in some cases. Two examples are treated in detail. The first one is the geodesic motion on a two–dimensional manifold of constant negative curvature. The second, a mathematical model, is based on Riemann's zeta function. Both have their own peculiarities: the somewhat unphysical character of the motion in the first, the purely speculative dynamical interpretation in the second. In spite of them, they both provide a concrete illustration of the general results obtained previously. The geodesic motion on a space of negative curvature has long served as a paradigm of classical and quantum chaotic motion [6, 27].

5.1 Geodesic flow on surfaces of constant negative curvature

We are interested in the geodesic motion on a two–dimensional hyperbolic geometry. In particular, we consider billiards on surfaces of constant negative curvature, the so–called Hadamard–Gutzwiller model. We will not enter into a detailed description of the classical and quantum motion in such surfaces, and refer the reader to the excellent introductory articles [27]. For these models Gutzwiller's trace formula is exact [28]. This allows to express the periodic orbit length spectrum in terms of the quantum eigenvalues [15]. We will here re–derived the formula for the length spectrum as an illustration of the general formalism based on the eigenvalues of the evolution operator and singularities of the Ruelle zeta.

Consider a compact and closed surface of area $\mathcal{A}$ on the two–dimensional Poincaré disk, constructed from a suitable bounded domain on which appropriate (periodic) boundary conditions have been defined. The corresponding classical geodesic motion includes a set p of primitive periodic orbits (closed geodesics on the compact surface). To start with, consider the corresponding Selberg zeta function [29],

$$Z_S(s) = \prod_p \prod_{n=0}^{\infty} \left(1 - \mathrm{e}^{-(s+n)\ell_p}\right) \ . \tag{35}$$

The first product is defined over the primitive periodic orbits p of length ℓ_p (in appropriate units). This function differs in its definition from the Smale and Ruelle zeta functions. $Z_S(s)$ is an entire function, with zeros located at [29]

a) $s = 1;\ \ g = 1$

b) $s = 1/2 \pm ip_\alpha;\ \ g = g_\alpha$

c) $s = 0;\ \ g = \mathcal{A}/(2\pi) + 1$

d) $s = -k,\ \ k = 1, 2, \ldots;\ \ g = (k+1)\mathcal{A}/(2\pi)$,

where $p_\alpha = \sqrt{E_\alpha - 1/4}$, $\alpha = 1, 2, \ldots$. The E_α are the (quantum) eigenvalues of the Laplace–Beltrami operator on the surface, and the p_α are the corresponding wavenumbers. The integer g denotes the multiplicity of the zeros. In case b) they depend on the degeneracy $g_\alpha \geq 1$ of the eigenvalue E_α. Because of topological constraints, $\mathcal{A}/(4\pi)$ is a positive integer.

The topological or Ruelle zeta function (4) is easily obtained from $Z_S(s)$. It is given by [29],

$$Z_R(s) = \prod_p \left(1 - \mathrm{e}^{-s\ell_p}\right)^{-1} = \frac{Z_S(s+1)}{Z_S(s)} \ . \tag{36}$$

The analytic structure of this meromorphic function directly follows from that of $Z_S(s)$ (see the left part of Fig.1),

a) pole at $s = 1;\ \ g_\eta = 1$

b) poles at $s = 1/2 \pm ip_\alpha;\ \ g_\eta = g_\alpha$

c) pole at $s = 0;\ \ g_\eta = \mathcal{A}/(2\pi)$

d) zeros at $s = -1/2 \pm ip_\alpha;\ \ g_\eta = -g_\alpha$

e) pole at $s = -1;\ \ g_\eta = \mathcal{A}/(2\pi) - 1$

f) poles at $s = -k,\ \ k = 2, 3, \ldots;\ \ g_\eta = \mathcal{A}/(2\pi)$.

The rightmost pole of this function is real and located at $s = 1$. This implies that $h_{\rm top} = 1$ in these systems. This pole is responsible for the leading asymptotic average growth of the number of orbits, and provides sub–leading corrections as well. There is an infinite number of other real poles with $q_\eta < h_{\rm top}$, which also provide sub–leading corrections to the average part. The complex zeros and poles, aligned here on two different vertical lines in the complex plane, contribute to the oscillatory part. There exist poles and zeros with arbitrarily large imaginary part, as generically required (cf §2).

Taking into account separately the contributions of the real and complex singularities of $Z_R(s)$, from Eq.(8) a formula is obtained for the length spectrum of the periodic orbits on a compact and closed hyperbolic surface (with the speed set to one, τ stands here for the length ℓ of the orbits),

$$\rho(\tau) = \overline{\rho}(\tau) + \widetilde{\rho}(\tau) \ , \qquad \tau = \text{length of orbits}, \tag{37}$$

with

$$\overline{\rho}(\tau) = \frac{1}{\tau} \sum_{m=1}^{m_c} \frac{\mu(m)}{m} \ {\rm e}^{\tau/m} \left[1 - {\rm e}^{-2\tau/m} + \frac{\mathcal{A}}{2\pi} \frac{{\rm e}^{-\tau/m}}{(1 - {\rm e}^{-\tau/m})} \right] \ , \tag{38}$$

and

$$\widetilde{\rho}(\tau) = \frac{4}{\tau} \sum_{m=1}^{m_c} \frac{\mu(m)}{m} \ \sinh(\tau/2m) \ \sum_\alpha g_\alpha \cos(p_\alpha \tau/m) \ . \tag{39}$$

Concerning the complex singularities, as mentioned before $Z_R(s)$ has two "critical" lines, one made of complex zeros (located at ${\rm Re}(s) = -1/2$), the other of complex poles (located at ${\rm Re}(s) = 1/2$). Their superposition produces the oscillatory part $\widetilde{\rho}(\tau)$. It is remarkable that the frequencies of the harmonic decomposition of the periodic orbit density are directly related to the quantum wavenumbers, a consequence of the fact that Selberg's trace formula is exact. We will come back to this point later on. The integrated version of Eq.(37) for the counting function, $N(\tau)$, was computed and analyzed in [15]. For a specific hyperbolic billiard, we have numerically checked that the deviations with respect to the leading order behavior $\overline{N}(\tau) \approx {\rm Ei}(\tau)$ observed in the data were well explained by the sub–leading corrections $m > 1$ of the pole $\eta = h_{\rm top} = 1$ in Eq.(16).

Notice that each of the terms in the series (38) diverges in the limit $\tau \to 0$. This divergence is due to the splitting of the density into two parts, $\overline{\rho}$ and $\widetilde{\rho}$, whereas the sum of the two terms is well behaved.

Consider now the spectral determinant or Smale zeta function (21), whose zeros are the eigenvalues of the evolution operator. Another peculiar feature of the Hadamard–Gutzwiller model is that all the periodic orbits have the same Lyapounov exponent (equal to one). M_p can be written as a two by two diagonal matrix whose diagonal elements are ${\rm e}^{\pm \ell_p}$. Therefore,

$$|\det(M_p^r - 1)| = 2 \ \left[\cosh(r\ell_p) - 1\right] \ . \tag{40}$$

Expanding the inverse of the determinant as $|\det(M_p^r - 1)|^{-1} = \sum_{k=1}^{\infty} k \; \mathrm{e}^{-rk\ell_p}$, and computing the sum over the repetitions r, from (21) and (40) we obtain

$$Z(s) = \prod_p \prod_{k=1}^{\infty} \left[1 - \mathrm{e}^{-(k-s)\ell_p}\right]^k \; . \tag{41}$$

Using the definition of the Ruelle and Selberg zeta functions, Eq.(41) can be re–expressed as

$$Z(s) = \prod_{k=1}^{\infty} Z_R^{-k}(k-s) = \prod_{k=1}^{\infty} \frac{Z_S^k(k-s)}{Z_S^k(k+1-s)} \; . \tag{42}$$

Further manipulations of the latter equation lead finally to

$$Z(s) = \prod_{k=1}^{\infty} Z_S(k-s) \; . \tag{43}$$

Since $Z_S(s)$ is entire, it follows from the last expression that $Z(s)$ is also an entire function. The analytic structure of $Z(s)$ follows from that of $Z_S(s)$, with zeros at (cf the right part of Fig.1),

a) $s = 0$; $g_\gamma = 1$

b) $s = k + 1/2 \pm i p_\alpha$, $k = 0, 1, 2, \ldots$; $g_\gamma = g_\alpha$

c) $s = k$, $k = 1, 2, \ldots$; $g_\gamma = k(k+1)\mathcal{A}/(4\pi) + 2$.

This distribution of zeros satisfies the "generic" requirements concerning the spectrum of the classical evolution operator of an hyperbolic system: (i) all zeros have $\mathrm{Re}(\gamma) \geq 0$, (ii) there is a simple pole at the origin (ergodic zero), and (iii) there are complex symmetric zeros with arbitrarily large imaginary part. Using Eq.(40), the density (28) is now expressed in terms of the classical resonances as,

$$\rho(\tau) = \frac{1}{\tau} \sum_{m=1}^{m_c} \frac{\mu(m)}{m} \; \mathrm{e}^{\tau/m} \; \left(1 - \mathrm{e}^{-\tau/m}\right)^2 \; \sum_\gamma g_\gamma \; \mathrm{e}^{-\gamma\tau/m} \; . \tag{44}$$

Using the locations and corresponding multiplicities of the zeros of $Z(s)$ given above, and separating the contributions of the real and complex zeros, it is easy to check that Eq.(44) is strictly equivalent to Eqs.(38) and (39). For the geodesic flow on a constant negative curvature, the exact density is thus recovered from Eq.(28), without any error. The reason for this is that in the present model the Lyapounov exponents are constant, independent of the orbit. It follows that Eq.(24) is exact for any period, not only asymptotically (cf Eq.(40)).

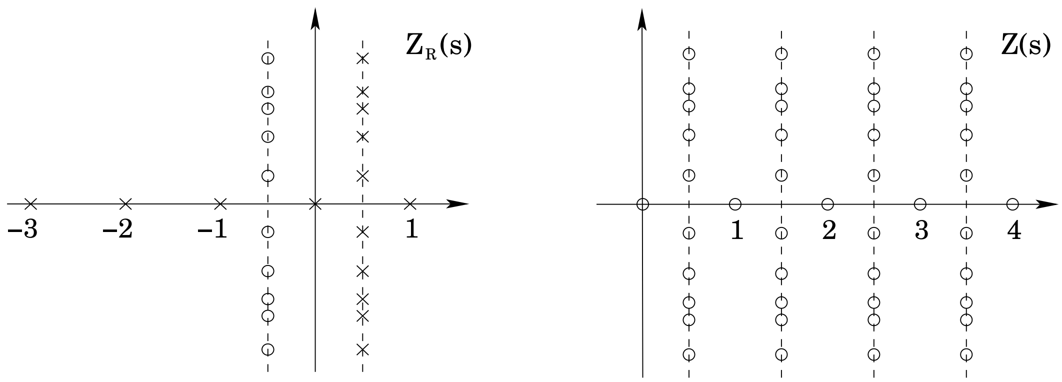

Figure 1: Analytic structure close to the origin of the Ruelle zeta function (left) and of the spectral determinant (Ruelle–Pollicott resonances, right) for a compact and closed billiard on a surface of constant negative curvature. Crosses are poles, circles are zeros. Multiplicities and exact positions are given in the text.

It is however interesting to remark that the two expressions for the density are obtained from functions with very different analytic structure. In particular, $Z_R(s)$ is meromorphic and has two "critical" lines, one made of an infinite number of zeros located at $\mathrm{Re}(s) = -1/2$, the other of an infinite number of poles located at $\mathrm{Re}(s) = 1/2$. In contrast, the spectral determinant is entire and, as complex zeros, has an infinite number of parallel replicas of the quantum spectrum located at $\mathrm{Re}(s) = k + 1/2$, $k = 0, 1, 2, \ldots$. The correct result (39) for the oscillatory part of the density is only recovered when the whole set of complex resonances of $Z(s)$ is taken into account. In contrast, the leading order asymptotic behavior $\tau \to \infty$ of the density is controlled by the zero at the origin and the complex resonances located at $\mathrm{Re}(s) = 1/2$.

The Hadamard–Gutzwiller model offers also the opportunity to analyze the accuracy of the approximation (34) of the Smale zeta $Z(s)$, which holds asymptotically. It is obtained by keeping only the first term $k = 1$ in Eq.(42). In that approximation, the analytic structure of $Z(s)$ consists now of a simple zero at the origin, complex zeros on the line $\mathrm{Re}(s) = 1/2$, complex poles on the line $\mathrm{Re}(s) = 3/2$, and zeros at $s = k, k = 1, 2, \ldots$. The analytic structure is thus well reproduced for $\mathrm{Re}(s) < 1$ (the zero at the origin and the lowest line of complex zeros located at $\mathrm{Re}(s) = 1/2$). The remaining structure is wrong (the degeneracies of the remaining real zeros is wrong, at $\mathrm{Re}(s) = 3/2$ it has a line of complex poles instead of zeros, the other lines of complex zeros are missing).

5.2 The Riemann zeta

Our second example is taken from analytic number theory, which has inspired several developments in the theory of dynamical systems [30]. The results we are going to present are based on a dynamical interpretation of the Riemann zeros and of the prime numbers. This interpretation is by no means necessary, but it is a useful one because it introduces the appropriate physical framework into the discussion, and therefore facilitates the comparison with dynamical systems. We therefore revisit here some well known formulas in number theory, viewed from the perspective of the present theory that connects the periodic orbits to the eigenvalues of the evolution operator.

The spectral interpretation of the Riemann zeros is based on the following identification. The imaginary part of each of the Riemann zeros is thought to be an eigenvalue of a quantum system with a classically chaotic limit with no time–reversal symmetry. An analysis, based on a semiclassical interpretation of a formula for the density of the critical zeros [31], shows that the set of prime numbers has to be identified to the set of periodic orbits of the (unknown) classical dynamics. The analysis leads to the following mapping,

$$\begin{array}{rcl} \text{primitive periodic orbits} & \rightarrow & \text{prime numbers } p \\ \text{period of the orbits } \ \tau_p & \rightarrow & \log p \\ \text{stability } |\det(M_p^r - 1)| & \rightarrow & p^r = \mathrm{e}^{r \log p} \ . \end{array} \tag{45}$$

The last correspondence implies that $h_{\rm top} = 1$ in this hypothetical dynamical system, that for simplicity we refer to as the Riemann dynamics.

We don't know the classical Hamiltonian behind the Riemann dynamics, but the information contained in (45) concerning the periodic orbits is enough to write down the trace of the corresponding classical evolution operator, Eq.(19). Using in the latter the correspondences (45), the trace is expressed as,

$$R(\tau) = \sum_p \sum_{r=1}^{\infty} \frac{\log^2 p}{p^r} \, \delta(\tau - r \log p) \ , \tag{46}$$

where the sum runs over the prime numbers p. To write down $R(\tau)$ in terms of the classical Ruelle–Pollicott resonances, we need to compute the corresponding spectral determinant, or Smale zeta function. The result, obtained from (21) and (45) is [13],

$$Z(s) = \zeta^{-1}(1-s) \ , \tag{47}$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function. Contrary to general expectations in bounded hyperbolic systems [21], $Z(s)$ is not entire for the Riemann dynamics, but meromorphic. The analytic structure of $Z(s)$ follows from that of $\zeta(s)$,

a) zero at $s = 0, \;\; g_\gamma = 1$

b) poles at $s = 1/2 \pm i \; t_\alpha, \;\; g_\gamma = -g_\alpha$

c) poles at $s = 1 + 2k, \;\; k = 1, 2, \ldots, \;\; g_\gamma = -1$.

Assuming the Riemann hypothesis, the t_α are real and define the position of the α'th zero of $\zeta(s)$ along the critical line, of multiplicity $g_\alpha \geq 1$. The pole of the Riemann zeta transforms into the ergodic zero of $Z(s)$. The position of the complex Ruelle–Pollicott resonances coincide here with the complex zeros of the Riemann zeta, but these are not zeros of $Z(s)$, but poles. Other real poles of $Z(s)$ are generated by the so–called trivial zeros of the Riemann zeta, now located on the real positive axis. The global properties of the distribution of the singularities of $Z(s)$ for the Riemann dynamics satisfy the general requirements of fully chaotic systems, with the important oddness related to the occurrence of some poles in place of zeros.

From (22) and the analytic structure of $Z(s)$ (assuming $g_\alpha = 1$), the function $R(\tau)$ may be written,

$$R(\tau) = 1 - \frac{\mathrm{e}^{-3\tau}}{1 - \mathrm{e}^{-2\tau}} - 2 \; \mathrm{e}^{-\tau/2} \; \sum_{\alpha=1}^{\infty} \cos(t_\alpha \tau) \; . \tag{48}$$

The "anomalous" minus signs that appear in the r.h.s. of this equation reflect, again, the presence of poles in $Z(s)$. The physical interpretation of these signs, and of the closely related minus sign that appears in front of the oscillatory part of the density of the Riemann zeros [31], is unclear for the moment, though an appealing possibility was suggested in [32].

The function $R(\tau)$, here interpreted as the trace of the classical evolution operator, is in fact a well known function in number theory,

$$R(\tau = \log x) = \frac{d\psi(x)}{dx} \; ,$$

where $\psi(x) = \sum_{n \leq x} \Lambda(n)$ ($\Lambda(n)$ is the Von–Mangoldt function) [18]. By inverting Eq.(46), as in §4, and using Eq.48), a formula for the density of the logarithm of the prime numbers is obtained, given by Eq.(28) with the appropriate identifications and replacements dictated by the table (45). The result coincides, up to a change of variable, with Riemann's formula. It can be integrated to obtain the counting function. These results stress the strong similarities that exist between number theory and the theory of dynamical systems.

The Ruelle zeta function (4) coincides with the Riemann zeta [30], $Z_R(s) = \prod_p (1 - p^{-s})^{-1} = \zeta(s)$. The approximation (34) is therefore exact in this case, since Eq.(32) is (cf Eq.(47)).

6 Concluding remarks

Two different explicit formulas for the density $\rho(\tau)$ of periods of the primitive periodic orbits of fully chaotic classical systems have been obtained. Both provide a harmonic decomposition of $\rho(\tau)$, where the complex zeros and poles (if any) of the corresponding zeta function are related to the elementary frequencies of the oscillatory terms, while the real ones contribute to the smooth part. In one case, the zeta function is the spectral determinant $Z(s)$. It is assumed to be an entire function; its zeros, denoted γ, are the eigenvalues of the classical evolution operator (usually called Ruelle–Pollicott resonances). In the second formulation, the relevant function is the Ruelle zeta $Z_R(s)$. In contrast to the spectral determinant, this function has a meromorphic extension in the complex plane. The relation between both approaches was discussed in some detail.

The zero of $Z(s)$ located at the origin (or, alternatively, the pole of $Z_R(s)$ located at $s = h_{\rm top}$) provides the leading average growth of the density of periods of periodic orbits, $\overline{\rho}(\tau) \overset{\tau\to\infty}{\sim} \mathrm{e}^{h_{\rm top}\tau}/\tau$. We found exponentially large sub–dominant corrections to the leading term that arise from the same zero (or pole), and that are responsible for the main deviations observed numerically in billiards in a surface of constant negative curvature.

The zero of $Z(s)$ at the origin reflects a generic property of hyperbolic systems, the existence of an equilibrium distribution described by the microcanonical measure. What about the rest of the spectrum? No generic statements are known, aside the fact that $\mathrm{Re}(\gamma) \geq 0$, that resonances with arbitrarily large imaginary part should exist, and that complex resonances come in symmetric pairs (cf §4). There are, however, some hints on what could probably be the generic structure, if any, of the low lying spectrum of $Z(s)$ (e.g., resonances with the smaller real part), that we would like to discuss now. This low–lying part of the spectrum describes the long time dynamics of the system. For simplicity, from now on we restrict the discussion to the case of billiards. We have in mind "generic" systems with a discrete spectrum of the classical evolution operator (exponential decay). We also restrict to ballistic systems (i.e., billiards whose shape produce a chaotic motion), and are excluding disordered systems.

In §5 we saw that for billiards on constant negative curvature the low–lying spectrum consists of the ergodic zero at $\gamma_0 = 0$, and of an infinite number of zeros located on the line $\mathrm{Re}(s) = h_{\rm top}/2$. This structure also follows for other systems from semiclassical arguments. For a fully hyperbolic billiard, Gutzwiller's trace formula for the density of quantum eigenvalues takes the form,

$$\rho_Q(t) = \sum_\alpha \delta(t - t_\alpha) \approx \overline{\rho}_Q(t) + \frac{1}{\pi}\sum_{p,r} \frac{\tau_p}{\sqrt{|\det(M_p^r - 1)|}} \cos(r t \tau_p) \ .$$

As in §5.1, τ_p denotes here the length of the periodic orbits p and t_α are the (eigen)

wavenumbers; $\overline{\rho}_Q(p)$ is the Weyl term, and we have neglected Maslov indices. Fourier inverting this formula with respect to t, and using the approximation (32), we obtain the following formula for the density of periods,

$$\rho(\tau) \approx \frac{\mathrm{e}^{h_{\text{top}}\tau}}{\tau} + \frac{2}{\tau}\ \mathrm{e}^{h_{\text{top}}\tau/2}\ \sum_{\alpha} \cos(t_\alpha \tau)\ . \tag{49}$$

This is precisely the leading order density that is obtained from (28) assuming that $Z(s)$ has a simple zero at the origin and complex zeros concentrated on the critical line $\text{Re}(s) = h_{\text{top}}/2$. Thus, the Hadamard–Gutzwiller model of §5 as well as the generalization Eq.(49) suggest that the generic low–lying spectrum of the classical evolution operator of fully chaotic billiards consists of a simple zero at the origin plus an infinite sequence (extending to arbitrary large imaginary parts) of complex symmetric zeros located on the line $\text{Re}(s) = h_{\text{top}}/2$. A corresponding structure follows for $Z_R(s)$ by transforming zeros into poles located at $h_{\text{top}} - \gamma$. This first hypothesis about the location of the complex zeros of $Z(s)$ is reminiscent to that of Riemann in number theory.

For the negative curvature model and in the inverse formula (49), the imaginary part of each Ruelle–Pollicott resonance located on the line $\text{Re}(s) = h_{\text{top}}/2$ coincides with a quantum wavenumber. This happens because of important non genericities of those systems: the corresponding semiclassical trace formula (the Selberg trace formula) is exact, and Maslov indices are zero. If in Eq.(49) the Maslov indices are not neglected (and correction terms are taken into account), the connection with the quantum wavenumbers is generically lost. The simplest effect of these phases would be to produce a shuffling of the resonances, without moving them out of the line $\text{Re}(s) = h_{\text{top}}/2$, and without changing their statistical properties. Some arguments in favor of this will be given below. Concerning the distribution of the eigenvalues of the classical evolution operator, our second guess is therefore that asymptotically (i.e., for resonances located far from the real axis) the statistical properties of the imaginary part of the Ruelle–Pollicott resonances located on the critical line $\text{Re}(s) = h_{\text{top}}/2$ coincide with those of the corresponding quantum (eigen) wavenumbers, which are random matrix like generically. This second hypothesis could be seen as an extension of the Bohigas–Giannoni–Schmit conjecture [33] to the statistical properties of the spectrum of the classical evolution operator. This way, the random matrix properties in fully chaotic systems would have a fully classical counterpart.

In semiclassical theories, the quantum correlations can be related to correlations acting among the actions of the periodic orbits [2]. In scaling systems like billiards, the action coincides, up to a constant factor, with the period (or length). Therefore action correlations are equivalent to period correlations. Since, as we have here shown, the density of periods of periodic orbits can be expressed in terms of the

eigenvalues of the classical evolution operator, it follows that the correlations between periods of periodic orbits can be expressed in terms of correlations acting among the Ruelle–Pollicott resonances. The quantum spectral correlations are thus mapped, via semiclassics, into classical spectral correlations. Through this connection, the RMT conjecture of the quantum fluctuations should have a classical counterpart, that applies to the fluctuation properties of the position in the complex plane of the Ruelle–Pollicott resonances. This gives some support to our second hypothesis.

The two previous conjectures determine the gross features of the low lying spectrum of the evolution operator in fully chaotic billiards and, as a consequence, of the long time behavior of the density and correlations of periodic orbits. The random matrix universality observed in quantum systems may have, by semiclassical arguments, a classical counterpart. The "ergodic" zero located at the origin certainly plays an important role [12]. We are here suggesting a clear and explicit additional link between the statistical properties of the classical and quantum spectrums, now involving the complex resonances.

To conclude, we briefly discuss the spectrum of the evolution operator for discrete maps, and show that important qualitative differences with respect to smooth flows occur. We have in mind area–preserving classically chaotic maps acting on two–dimensional phase spaces, like for example the kicked Harper or the kicked top. The time now takes only discrete values, $\tau = n$, $n = 1, 2, 3, \ldots$ (in some arbitrary units), and the set of possible lengths of periods of periodic orbits is trivial, just integers. The "return probability" or trace of the evolution operator is still expressed as a sum over the periodic points [4],

$$R(n) = \mathrm{tr}\mathcal{L}^n = \sum_p \sum_{r=1}^{\infty} \frac{n_p}{|\det(M_p^r - 1)|}\ \delta_{n, r n_p} = \sum_\gamma g_\gamma\ \mathrm{e}^{-\gamma n}\ , \tag{50}$$

where n_p is the (integer) period of the periodic orbit p, and the γ's are the Ruelle–Pollicott resonances. The latter are the zeros of (21), making the replacement $\tau_p \to n_p$. Inverting Eq.(50) as in §4, a formula follows for the number of primitive periodic orbits of period n,

$$\mathcal{N}(n) = \frac{1}{n} \sum_{m/n} \mu(m)\ |\det(M_{n/m} - 1)|\ \sum_\gamma g_\gamma\ \mathrm{e}^{-\gamma n/m}\ . \tag{51}$$

As in §4, we have ignored the fluctuations of the factor $|\det(M_\ell - 1)|$ which may occur at short periods. An alternative formula, equivalent to (16), is obtained from the corresponding Ruelle zeta, $\mathcal{N}(n) = n^{-1} \sum_{m/n} \mu(m)\ \sum_\eta g_\eta\ \mathrm{e}^{\eta n/m}$, where the η's are the poles and zeros of Eq.(4) with the replacement mentioned above. Since the time $\tau = n$ changes by unit steps, the smallest scale over which temporal

variations can occur is one. It follows from Eq.(50) that the complex resonances satisfy $-\pi < \mathrm{Im}(\gamma) \leq \pi$. Therefore, the natural variable to analyze discrete maps is not s, but rather $z = \mathrm{e}^{-s}$. By this transformation, the ergodic zero $\gamma_0 = 0$ is located at $z = 1$, and other real and complex Ruelle–Pollicott resonances with $\mathrm{Re}(\gamma) > 0$ lie inside the unit disk $|z| \leq 1$.

Substantial differences are expected between the structure of the spectrum of $Z(s)$ and $Z_R(s)$ for chaotic maps with respect to that of chaotic continuous flows, and in particular with respect to chaotic billiards. The main reason for that is the simplicity of the spectrum of periods of periodic orbits in the case of maps. Since that spectrum is made of integers, the only non–trivial information carried by Eqs.(50) and (51) concerns variations in the stability factors and number of orbits of a given period (e.g., average coarse–grained properties). This is in contrast with continuous chaotic billiards, where a much more subtle and delicate information is encoded in the spectrum of resonances, namely the nontrivial distribution of the periods of the orbits. In maps, that distribution collapses to a simple and highly degenerate spectrum. The simplicity of the average properties encoded in Eq.(50), without fine–grained structure, implies a simpler spectrum of resonances. In particular, no concentration of resonances over a "critical" line is expected to occur (this would be a critical circle inside the unit disk in the z variable), since its presence is associated to a harmonic decomposition of the distribution of periods in $R(\tau)$, whereas $R(n)$ strictly tends to a constant for long times. In hyperbolic maps, isolated resonances, without any special structure in the radial direction, are therefore expected generically inside the unit circle. No particular connection with random matrix theory is therefore established concerning the statistical properties of the resonances of chaotic maps. This picture seems to be confirmed by recent numerical simulations [14].

Semiclassically, the reason for the important differences with respect to chaotic billiards is also clear. Random matrix requires correlations between actions of periodic orbits. In scaling systems like billiards, the action of an orbit is proportional to its length (or period). Therefore action correlations translate into length (or period) correlations, which in turn, through Eq.(28), translate into resonance correlations. In maps, the set of possible periods (which is trivial) is not related to that of actions (which is nontrivial). Therefore, the spectrum of the evolution operator, which is closely related to the spectrum of periods of the periodic orbits, looses its connection with random matrix theory.

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